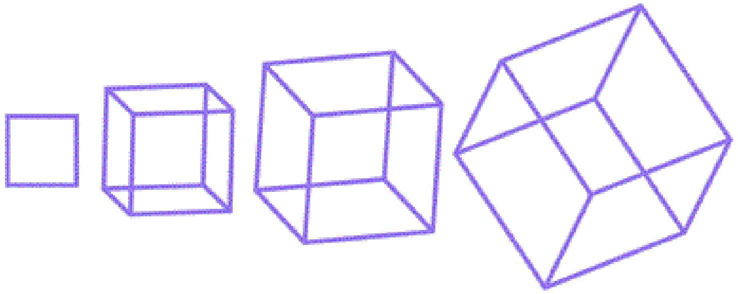
Czech University of Life Sciences Prague
Faculty of Environmental Sciences



Stochastic estimation of Nash equilibrium in geographic and other spatial data

MAREK VACH

DEPARTMENT OF WATER RESOURCES AND ENVIRONMENTAL MODELING

FACULTY OF ENVIRONMENTAL SCIENCES

CZECH UNIVERSITY OF LIFE SCIENCES PRAGUE

Introduction

Player 1 \ Player 2	a	b	c
a	3, 1	3, 2	1, 4
b	4, 2	2, 1	0, 1
c	0, 0	2, 4	5, 3

- The Nash equilibrium concept (Nash 1951) in linear games is an effective tool that is very often applied in many, not only economic contexts.
- Any bimatrix or multidimensional regular configuration containing in its elements any pay off values corresponding to the number of dimensions can be interpreted as a game.
- Each game has Nash equilibrium (NE) in pure or mixed strategies.

Introduction

Game theory involving NE concept has been developed in many contexts, the most basic can be:

- **Stochastic games** (Shapley 1953) - games that consist of a sequence of states. The pay off in each state is defined by the transition probability function dependent on the previous state and the players actions
- **Games with Incomplete Information** - **Bayesian games** (Harsanyi 1967) - from a single player perspective, the uncertainty about the characteristics of other players is modeled as the probability distribution of their pay off functions
- **Purification theoreme** (Harsanyi 1973) - Almost all Nash equilibria in mixed strategies can be reconceptualized as close approximations of Bayes Nash equilibrium in pure strategies
- **Evolutionary games** (Maynard Smith, Price 1973) - A suitable gaming model defines the system of ODE - replicator dynamics that simulates evolutionary dynamics leading to the evolutionarily stable strategy (ESS).

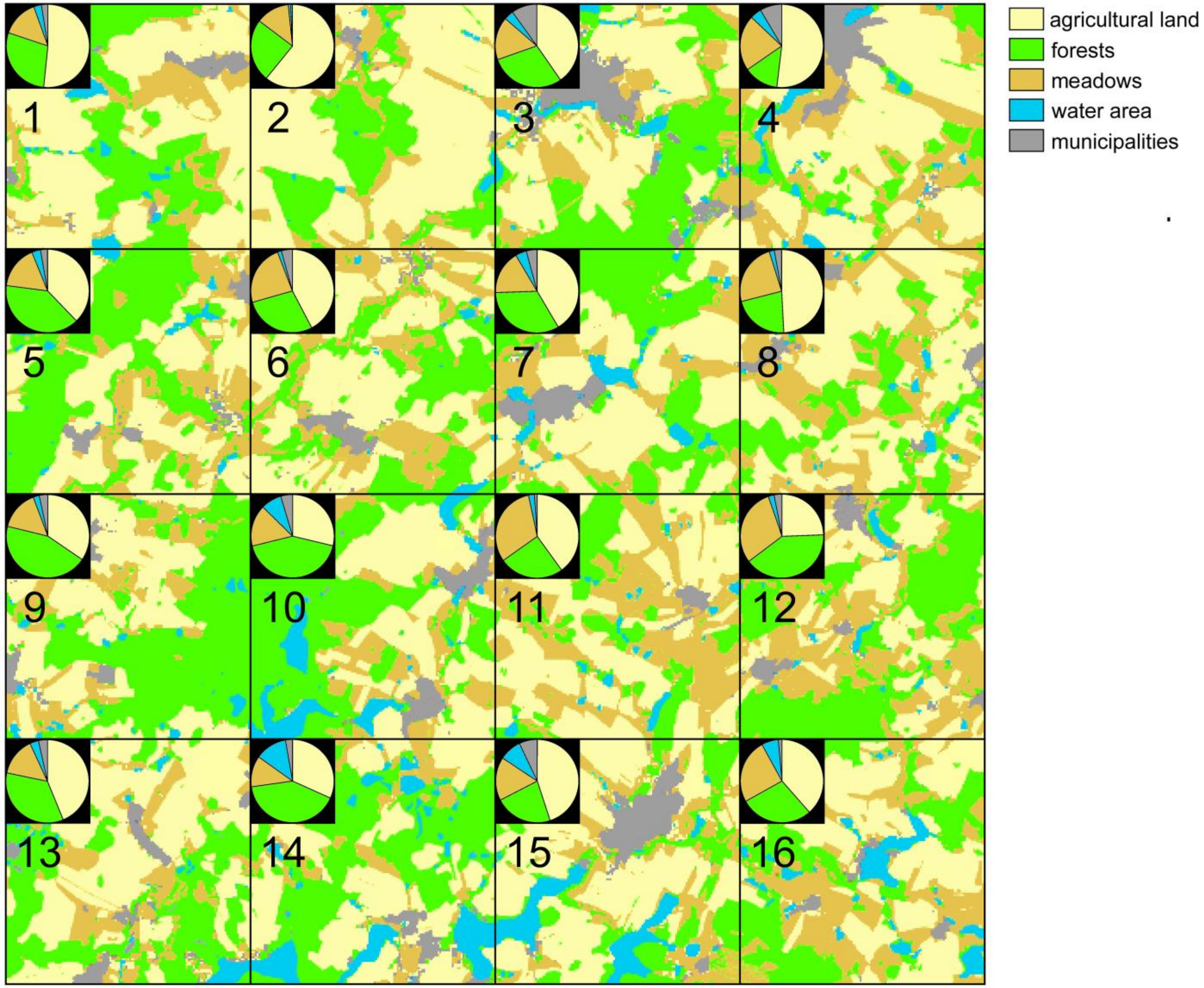
Research Questions

Normal-form game is at least:

1. A set of **players** - $P = \{1, 2, \ldots, N\}$
2. A set of their possible primarily defined sets of **strategies** - $\mathbf{S} = \{S_1, S_2, \ldots, S_N\}$
3. A set of **pay off** functions $\mathbf{F} = \{F_1, F_2, \ldots, F_N\}$, $F : S_1 \times S_2 \times \ldots \times S_N$

Is it possible to use the game concept to determine the distribution of NE in spatial data?

Any partial areas can be defined in spatial data. Each sub-territory is characterized by different parameter values that can be chosen in spatial data. These parameters can be considered as **players** and their values can be taken as **pay off**.



Research Questions

But spatial data does not provide any information about the primary **strategies** of individual players to define the game.

There are no strategies of players to create the game, where to take them?

The game concept is primarily defined in any context and application:

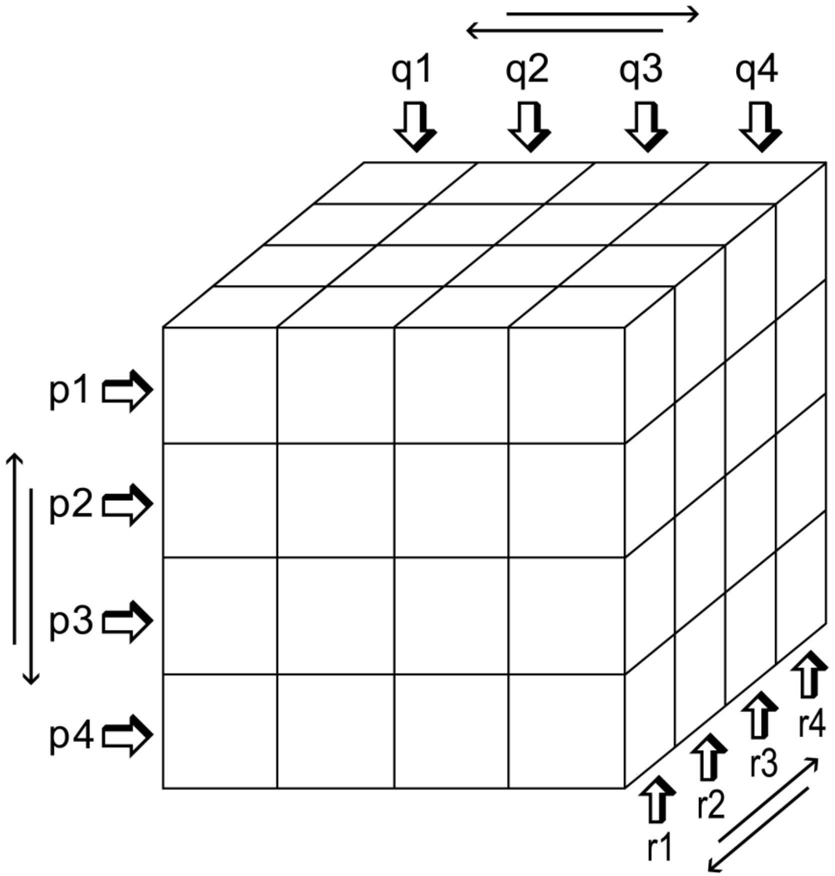
Strategies → Pay off → NE (or ESS)

Primarily, strategies are defined and pay off functions determine the expected outcome - which **follows** the strategy choice.

For the possibility of applying the NE concept to spatial data, the causality (strategies → pay off) **must be reversed**.
However, specific strategies defining game configuration cannot be determined from nothing.

The only option is a stochastic approach that takes into account all possible createable games (with different NE) representing all possible formal unspecified strategies.

Proposed approach



Situations with different values of chosen characteristics defined in spatial data (sub-territory etc.) are considered as the configuration of pay off vectors of dimension *N* in linear games in basic form.

These symmetric configurations of *N* interacting entities (ie characteristics in situations) are compiled as a **combination** of all *K* situations depending on *N*, *K* and the number of formal strategies *A* considered for one entity, the number is: $\binom{K}{A^N}$

Individual games (in which NE is found) are formed by **permutations** of elements of symmetrical configuration.

Strategies of interacting entities are not specified - **distribution of NE is assigned to situations** (to elements of game configuration).

Therefore, the **order of columns or rows** of derived game configurations **does not matter** and the number of **effective permutations** is: $$\frac{(A^N)!}{A!^N}$$

It is easy to prove that **symmetrical** configurations have the greatest number of effective permutations.

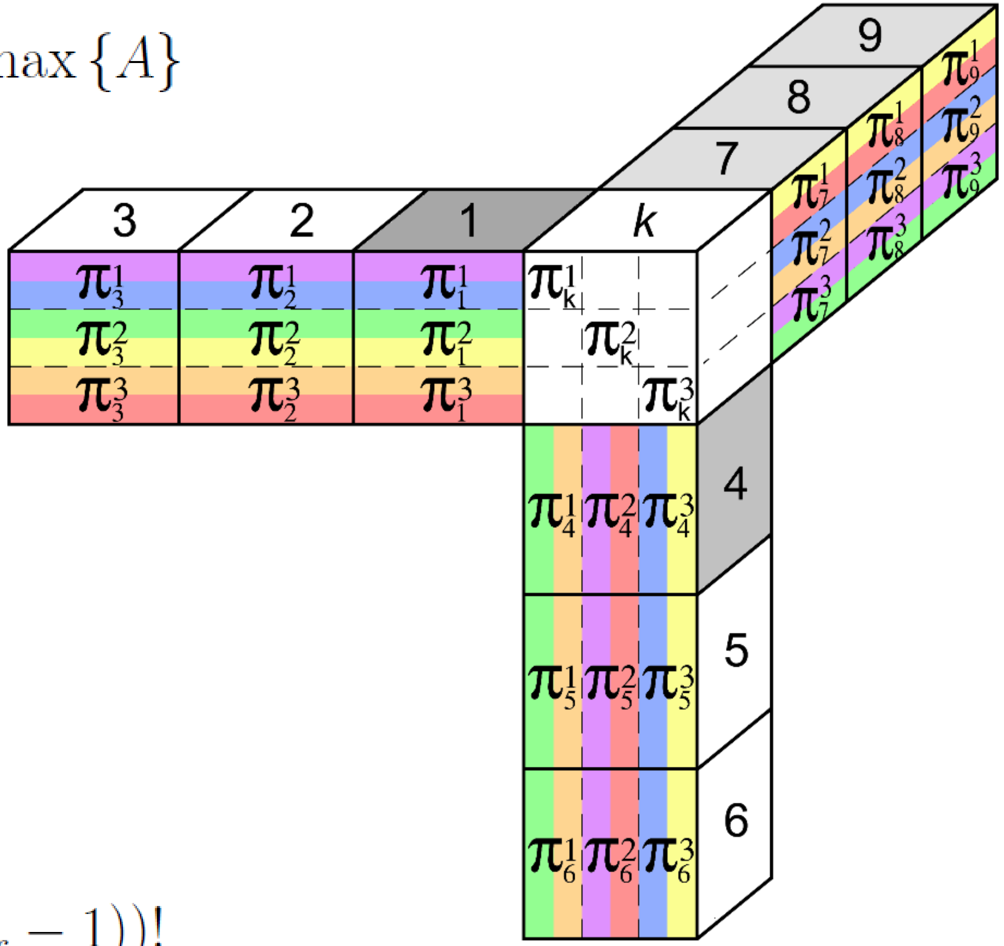
Proposed approach - Calculate the probability of NE in pure strategies

For K situations, game configurations with A rows and columns can be considered: $A \in \mathbb{N}; 2 \leq K^{1/N}, A_{max} = \max\{A\}$

To evaluate NE in pure formal strategies, it is appropriate to build on the largest game configuration available for K situations - with A_{max} rows and columns.

Calculating the NE probability in the k-th situation for N interacting entities can be based on the evaluation of ***N*×(*A*$_{max}$-1)-tuples** representing the pay off vectors of the row and column intersecting in k **without need of evaluation the entire game configuration**. The number of combinations / permutations in $N\times(A_{max}$-1)-tuple is:

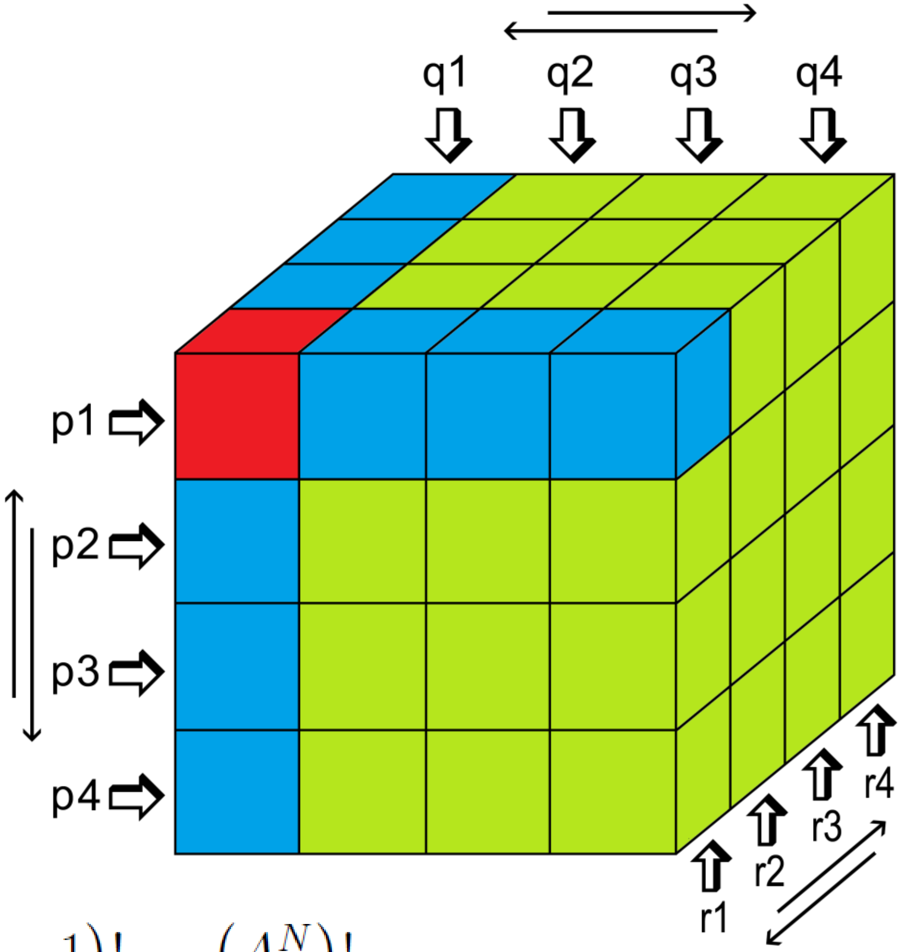
$$T_k^{sum} = \prod_{n=1}^{N} \binom{(N-n+1)(A_{max}-1)-1}{A_{max}-2} \cdot N! = \frac{(N(A_{max}-1))!}{(A_{max}-2)!^N N! (A_{max}-1)^N} \cdot N! = \frac{(N(A_{max}-1))!}{(A_{max}-1)!^N}$$



Proposed approach - Calculate the probability of NE in mixed strategies

Evaluation of NE probability for defined *K* situations in mixed formal strategies includes calculation of NE for symmetric game configurations with row / column length in the interval [2, A_{max}].

The found NE represents the total value of 1 for the interacting entity - the calculated shares of the strategies assigned to each situation are divided by the number of situations belong to each strategy – A^{N-1}.



The total number of evaluated effective matrix combinations / permutations can be expressed:

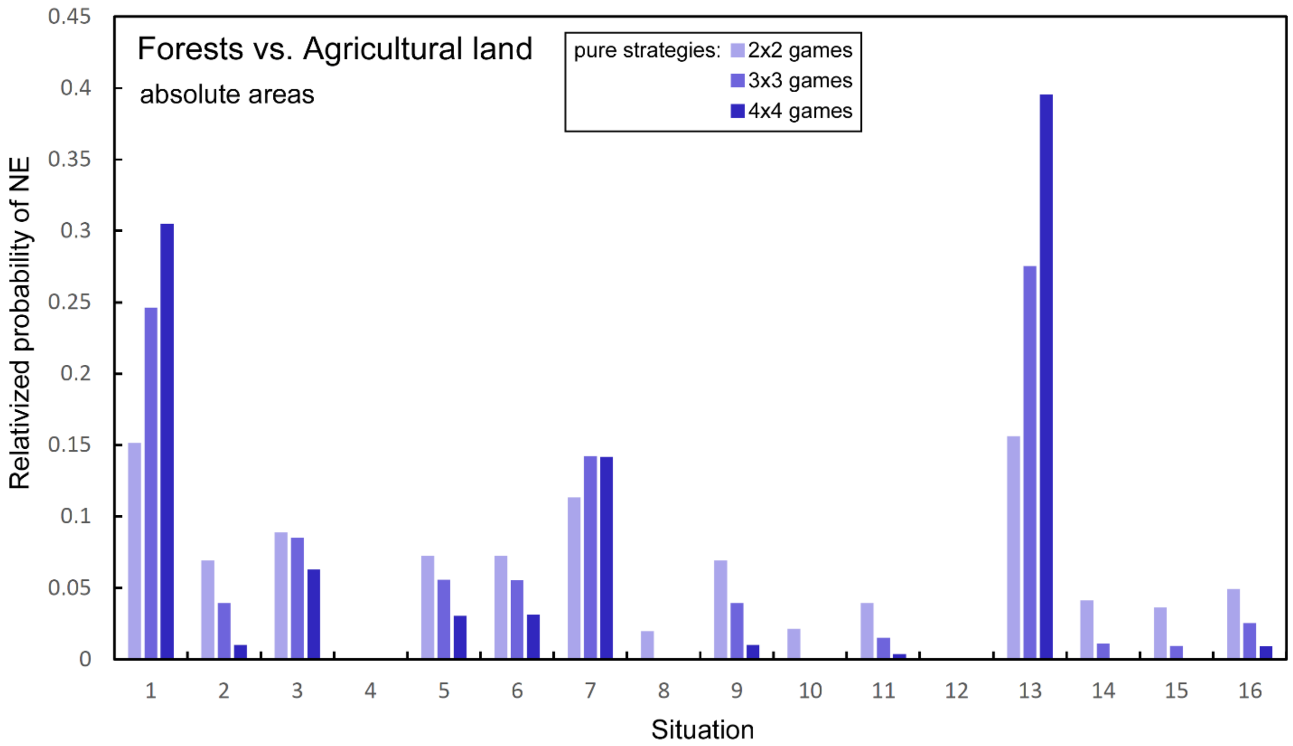
$$T_A^{mix} = \frac{(A^N-1)!}{(N(A-1))!}\prod_{n}^{N}\binom{(N-n+1)(A-1)}{(A-1)} = \frac{(A^N-1)!}{(A-1)!^N} = \frac{(A^N)!}{A!^N}$$

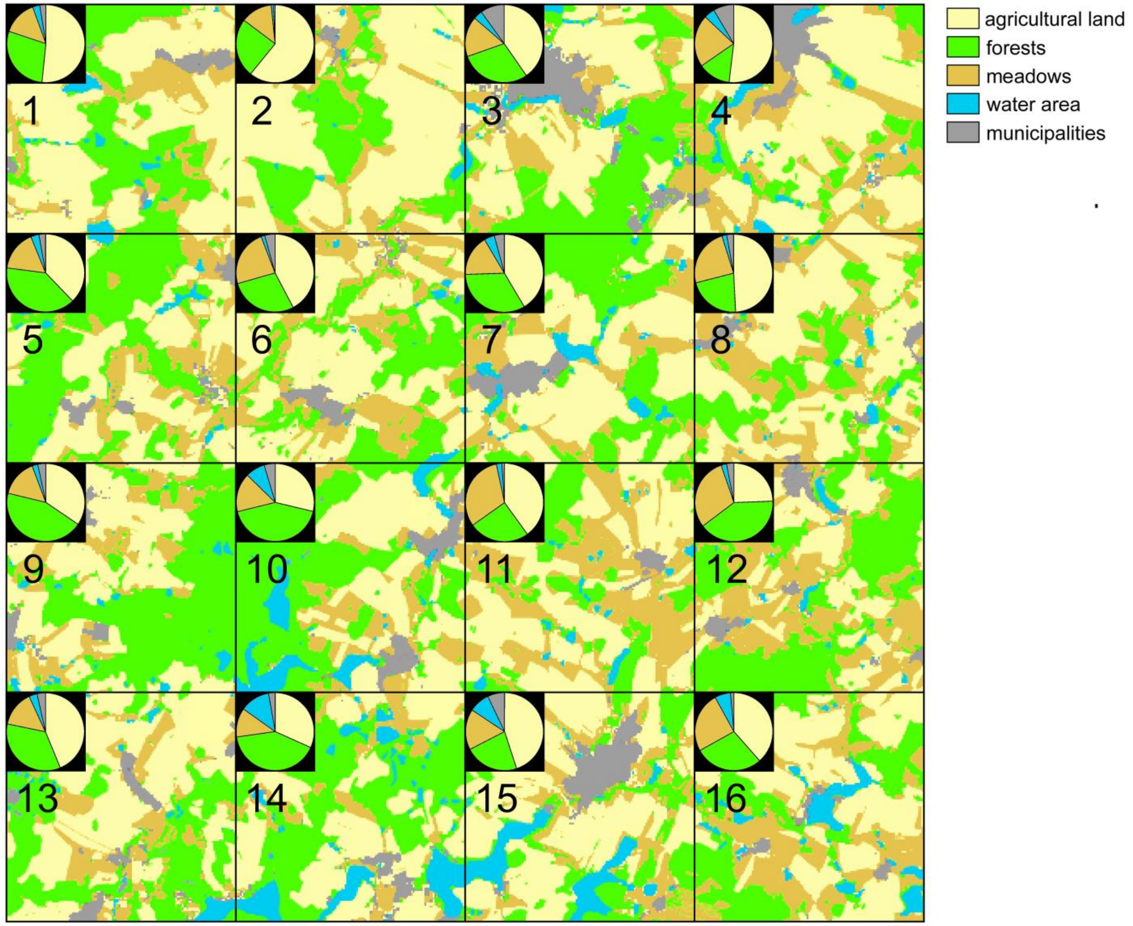
Proposed approach - Calculate the probability of NE

Each permutation of a N-dimensional game configuration is a game that has at least one NE in pure or mixed strategies (Nash 1951), so that for any interacting entity $n \in [1, N]$:

$$\sum_{k=1}^{K} \left(p_k^{pure} + \sum_{A=2}^{A_{max}} p_A^{mix}{}_k^n \right) \geq 1$$

Example of application of the proposed approach - results



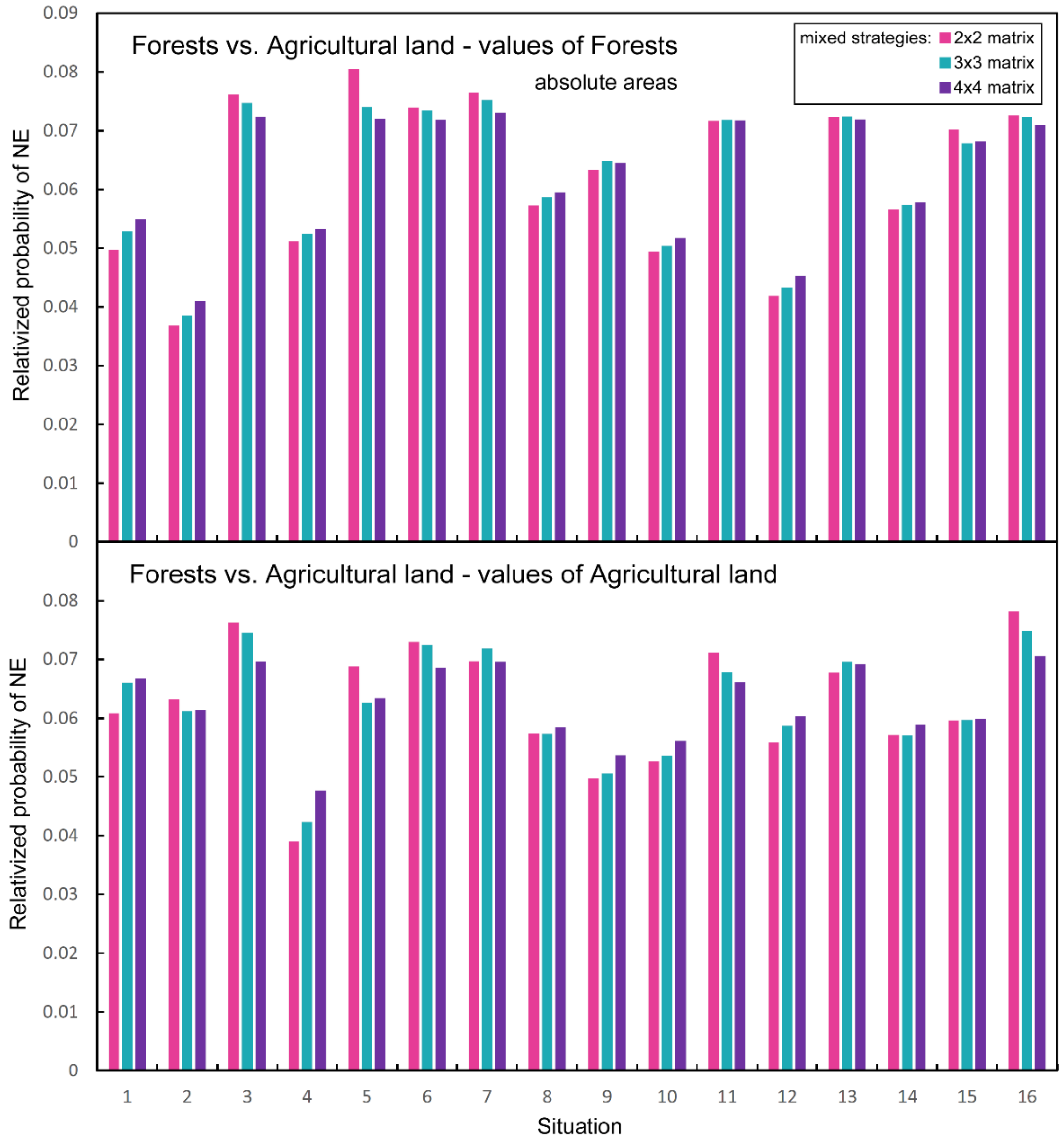


Pure formal strategies – Forests vs. Agricultural land

The size of the game matrix used does not affect the order of occurrence probability of NE calculated in pure strategies.

Increasing the number of formal strategies in the symmetric game matrix leads to the highlighting of the differences between the sub-territories (between the situations) and thus to more clear results.

Example of application of the proposed approach - results

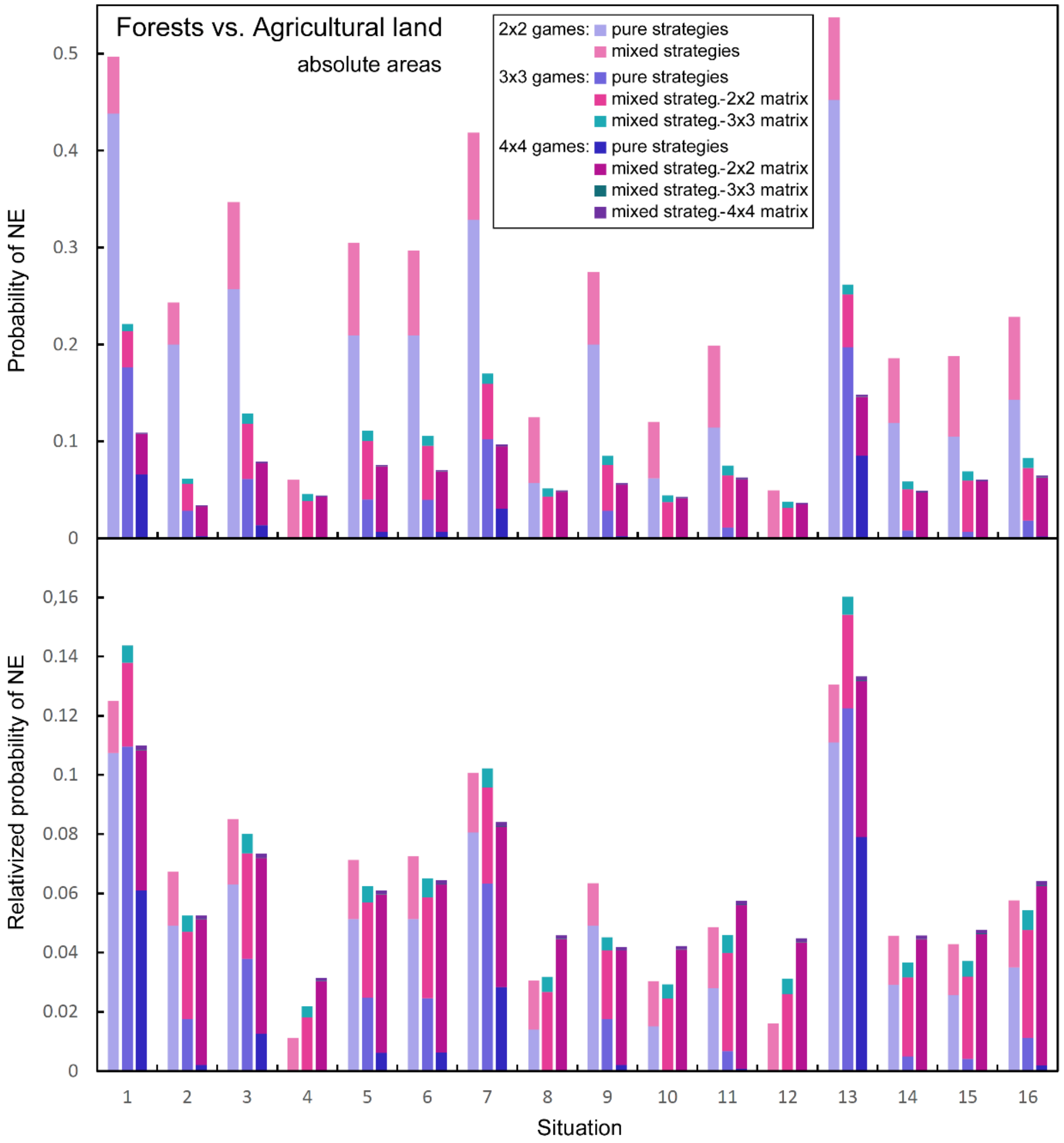


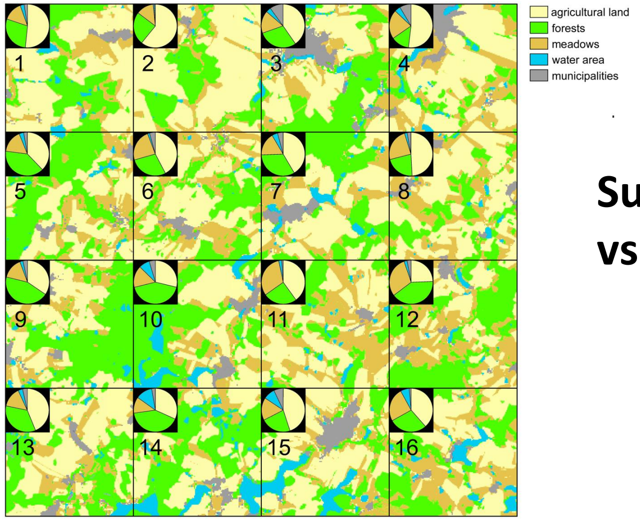
Mixed formal strategies – Forests vs. Agricultural land

The size of the game matrix used does not affect the order of occurrence probability of NE calculated in mixed strategies.

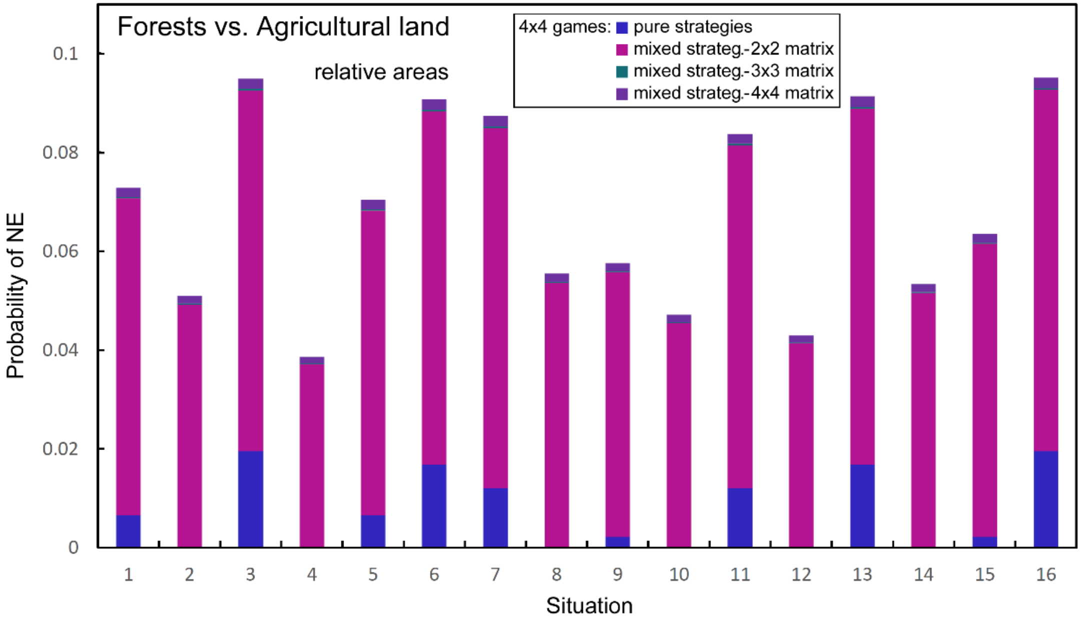
The NE probability values in mixed strategies, calculated for absolute areas, differ to some extent in both interacting entities - Forests vs. Agricultural land.

Example of application of the proposed approach - results

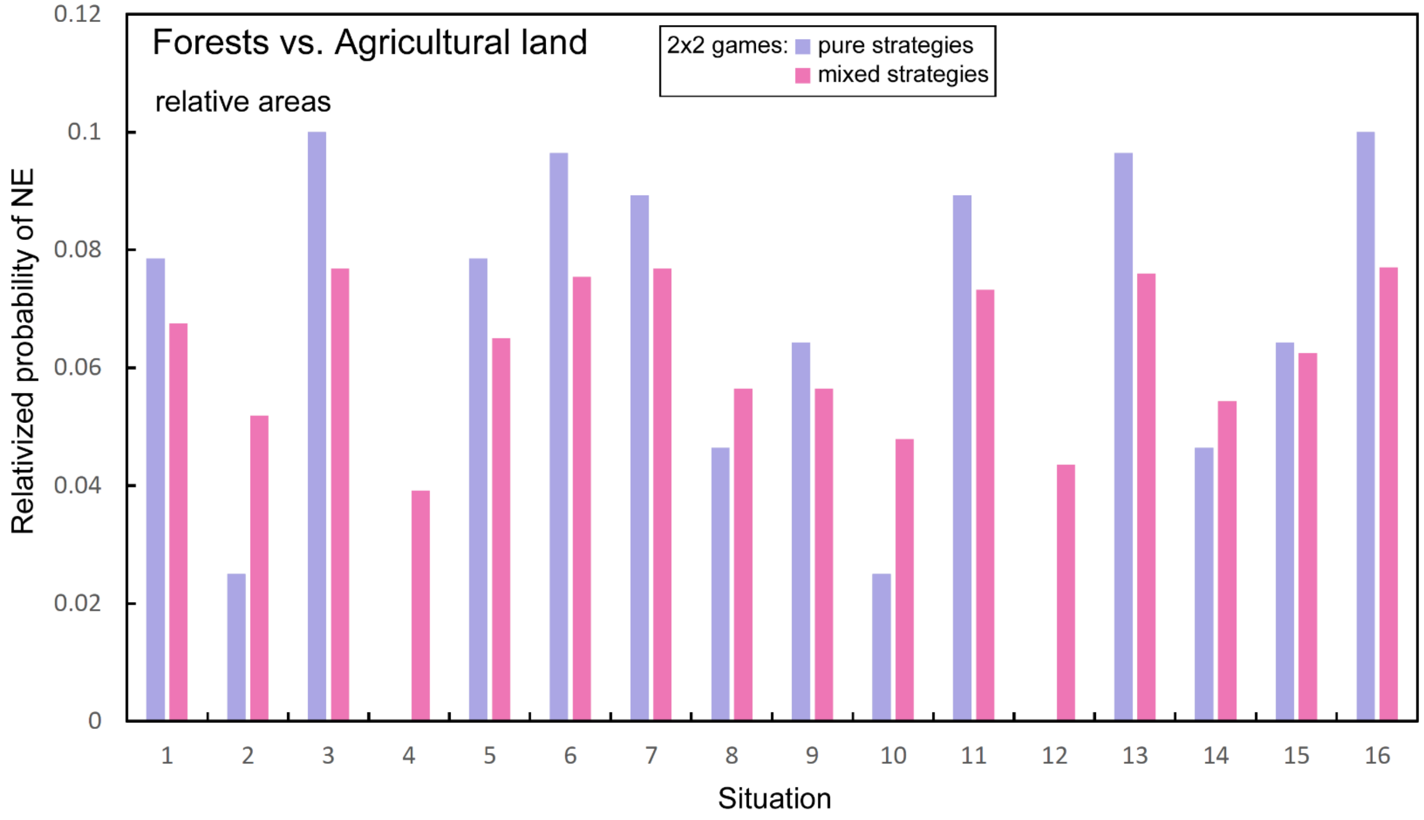




Summary values – Forests vs. Agricultural land



Example of application of the proposed approach - results

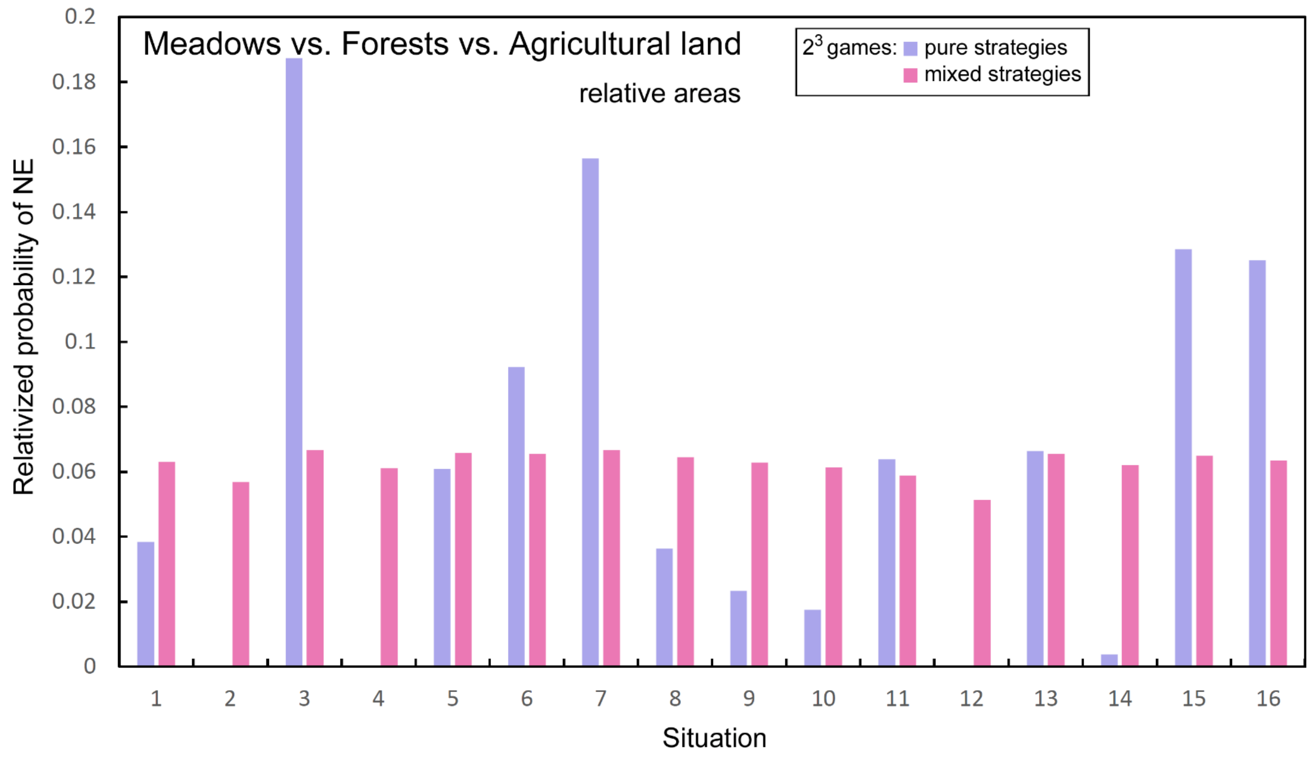


Comparison of NE probability trend in pure and mixed strategies for 2x2 games, calculation for **relative areas of Forests and Agricultural land.**

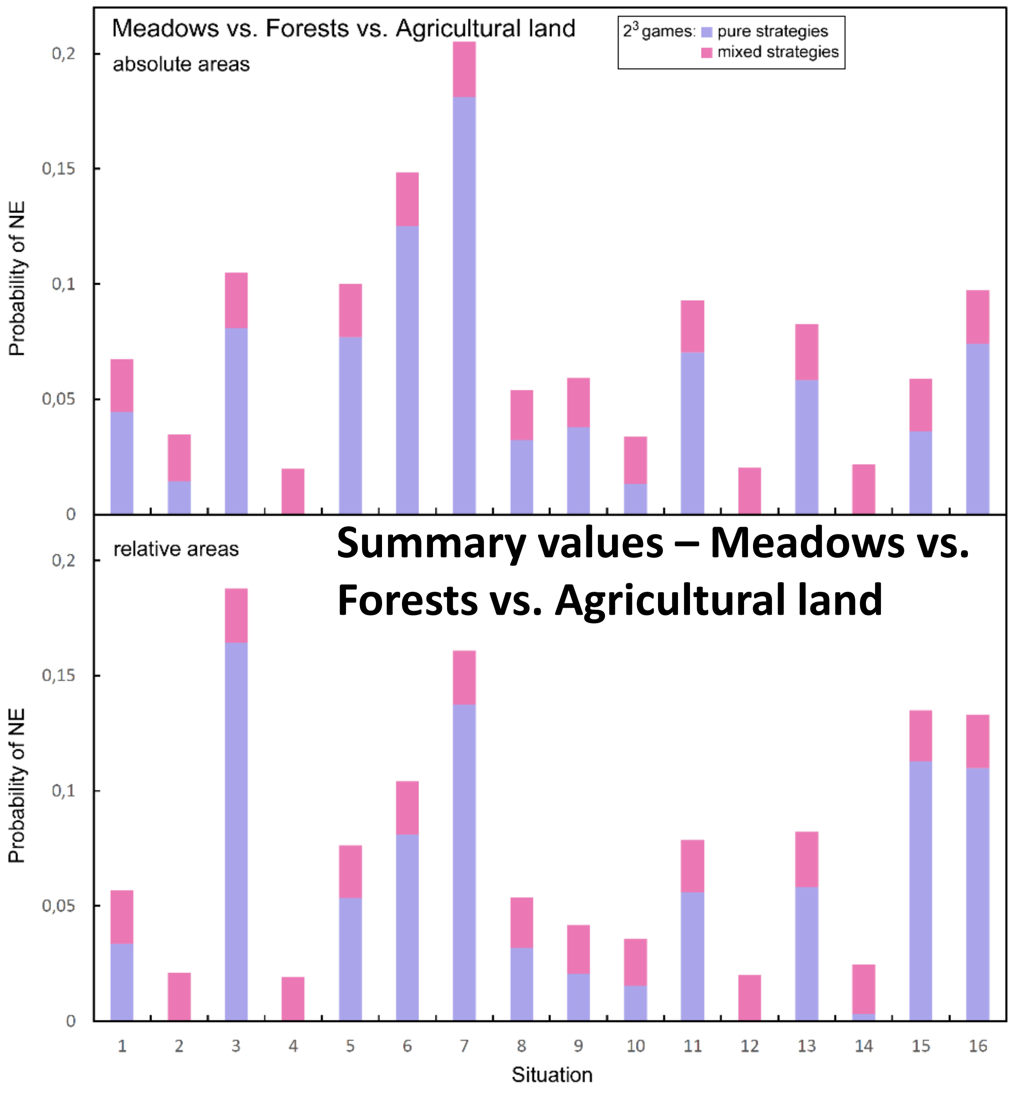
The trend is similar for pure and mixed strategies

Example of application of the proposed approach - results

Comparison of NE probability trend in **pure and mixed strategies**, calculation for relative areas of 3 interacting entities **– Meadows, Forests and Agricultural land.**

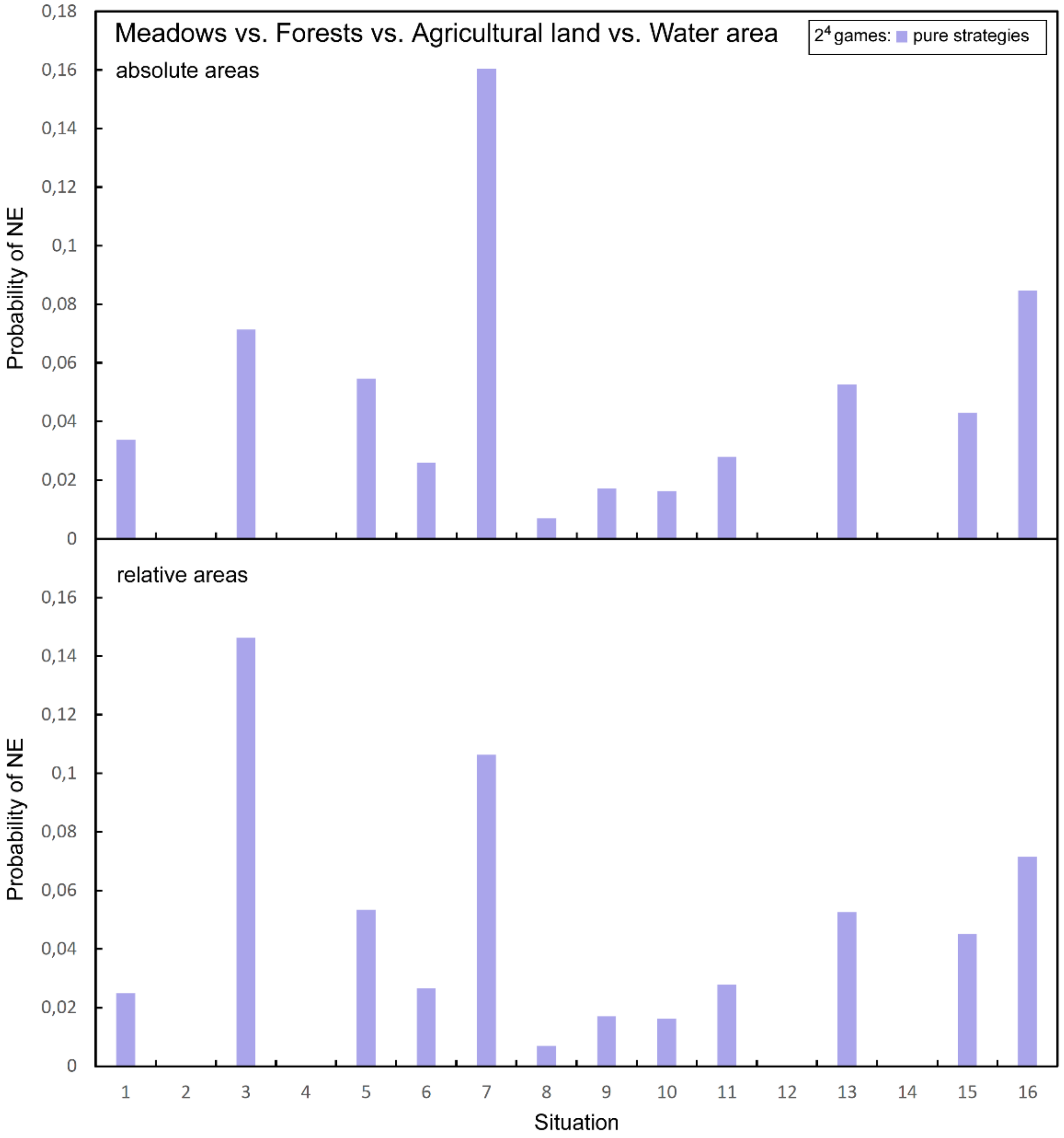


The significance of NE probability values in mixed formal strategies is virtually negligible.

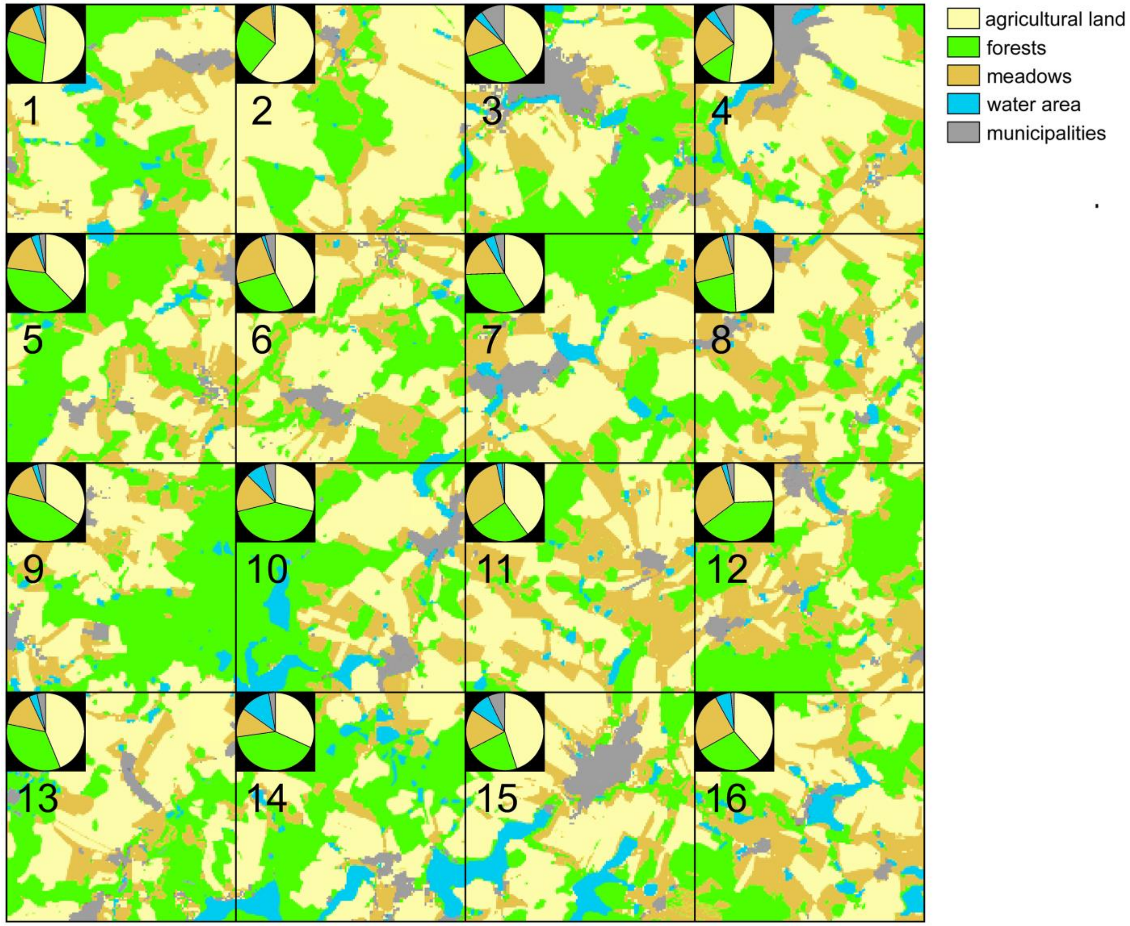


Summary values – Meadows vs. Forests vs. Agricultural land

Example of application of the proposed approach - results



Meadows vs. Forests vs. Agricultural land vs. Water area - NE probabilities in pure strategies calculated for absolute and relative areas.



Discussion and conclusions

The proposed approach can be categorized as - **„Games“ with stochastically derived formal strategies**

It could be said that the approach used where the permutations of pay off configuration generate formal unspecified strategies virtually denies the very principle of the game.

However, these **permutations generate games** - any bimatrix or multidimensional regular configuration containing in its elements any pay off values corresponding to the number of dimensions can be interpreted as a game.

So these are **games**, but **without a specific interpretation - nothing is known about this games**. But it is certain that **all possible games** of interacting entities - players **are evaluated**.

Discussion and conclusions

The example of applying the proposed procedure to a model unspecified land use map represents an effective analysis of the properties of this type of evaluation. Showed that:

- **NE probability** distribution is **determined by** contributions found in **pure strategies**, NE contributions in **mixed** strategies are **not significant**, especially for more interacting entities.
- Thus, the proposed approach can be used to evaluate larger data sets of representative multiple interacting entities (NE probability is calculated only in pure strategies).

The possibilities of the proposed approach are applicable to spatial data sets with parameters for which the NE probability can be related to the local stability of landscape and others ecosystem processes, which is important in many contexts.

The proposed approach is used in a simplified form in the published work:

Vach M., Vachova P. Stochastic Identification of Stability of Competitive Interactions in Ecosystems. Plos One. 2016;11(5): e0155023.

References (only selected basic works):

Nash J. Non-Cooperative Games. Ann Math. 1951;54(2): p.286.

Shapley LS. Stochastic Games. Proc Natl Acad Sci. Natl Acad Sciences; 1953 Oct 1;39(10): 1095–100.

Harsanyi JC. Games with Incomplete Information Played by “Bayesian” Players, I-III Part I. The Basic Model. Manage Sci. 14(3,5,7): (159–82) (320–34) (486–502).

Harsanyi JC. Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points. Int J Game Theory. 1973 Dec;2(1): 1–23.

Maynard Smith J, Price GR. The Logic of Animal Conflict. Nature. Nature publishing group; 1973 Nov 2;246(5427): 15–8.